

INTRODUCTION

In our earlier classes, we have been multiplying a number by itself. $2 \times 2, 3 \times 3 \times 3, 5 \times 5 \times 5 \times 5, a \times a, x \times x \times x$ are examples of repeated multiplication. Think of a situation when a number is multiplied by itself say 20 times or 30 times. Is it not time consuming to write $3 \times 3 \times 3 \dots 20$ times or $4 \times 4 \times 4 \times \dots 30$ times? To overcome this difficulty, we study the concept of indices. In this chapter, we shall study the concept of exponents (also known as indices), positive indices, negative indices or zero index and fractional indices. We shall also learn laws of exponents and shall use them in problems of simplification.

6.1 INDICES

We know that

$$\begin{aligned} 2 \times 2 &= 2^2 \\ 2 \times 2 \times 2 &= 2^3 \\ \frac{2}{3} \times \frac{2}{3} &= \left(\frac{2}{3}\right)^2 \\ \frac{5}{11} \times \frac{5}{11} \times \frac{5}{11} \times \frac{5}{11} &= \left(\frac{5}{11}\right)^4 \end{aligned}$$

Similarly,

$$\begin{aligned} x \times x &= x^2 \\ x \times x \times x &= x^3 \\ x \times x \times x \times x &= x^4 \\ \dots &\quad \dots &\quad \dots \\ \underbrace{x \times x \times x \times \dots \times x}_{n \text{ times}} &= x^n \end{aligned}$$

We read $x^2, x^3, x^4, \dots, x^n$ as **second, third, fourth, ..., nth power of x**.

Thus, in x^2 , index or power of x is 2. Alternatively we say that **exponent of x is 2** x is called the **base**.

Remarks

1. We also read x^n as x raised to the n th power
2. Plural of 'index' is indices.

Definition

An exponent or index is a number which indicates how many times another number, the base, is being used as a repeated factor.

6.2 LAWS OF INDICES

We give below the laws of indices :

- (i) $x^m \times x^n = x^{m+n}$, where m and n are +ve integers and x is a real number.
- (ii) $(x^m)^n = x^{mn}$, where m and n are +ve integers and x is a real number.
- (iii) $x^m \div x^n = x^{m-n}$, where m and n are +ve integers ($m > n$) and x is a non zero real number.

(iv) $(xy)^m = x^m \cdot y^m$ where m is a +ve integer and x, y are real numbers.

(v) $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$, where m is a +ve integer and x, y are real numbers, $y \neq 0$

These laws are called **Fundamental Laws of Indices**.

Proof.

(i)

$$\begin{aligned} x^m \times x^n &= \underbrace{x \times x \times x \times \dots}_{m \text{ factors}} \times \underbrace{x \times x \times x \times \dots}_{n \text{ factors}} \\ &= \underbrace{x \times x \times x \times \dots}_{(m+n) \text{ factors}} \\ &= x^{m+n} \end{aligned}$$

Hence,

$$x^m \times x^n = x^{m+n}$$

(ii)

$$\begin{aligned} (x^m)^n &= \underbrace{x^m \times x^m \times x^m \times \dots}_{n \text{ factors}} \\ &= x^{m+m+m+\dots n \text{ times}} \\ &= x^{mn} \end{aligned}$$

Hence,

$$(x^m)^n = x^{mn}$$

(iii)

$$\begin{aligned} x^m \div x^n &= \frac{x^m}{x^n} \\ &= \frac{x \times x \times x \times \dots m \text{ factors}}{x \times x \times x \times \dots n \text{ factors}} \\ &= x \times x \times x \times \dots (m-n) \text{ factors} \quad \dots [\because m > n] \\ &= x^{m-n} \end{aligned}$$

Hence,

$$x^m \div x^n = x^{m-n}$$

Note. If $m < n$, then $x^m \div x^n = \frac{x^m}{x^n} = \frac{1}{x^{n-m}}$

(iv)

$$\begin{aligned} (xy)^m &= \underbrace{xy \times xy \times xy \times \dots}_{m \text{ times}} \\ &= \underbrace{x \times x \times x \times \dots}_{m \text{ times}} \times \underbrace{y \times y \times y \times \dots}_{m \text{ times}} \\ &= x^m \times y^m \end{aligned}$$

Hence,

$$(xy)^m = x^m \times y^m$$

(v)

$$\begin{aligned} \left(\frac{x}{y}\right)^m &= \underbrace{\frac{x}{y} \times \frac{x}{y} \times \frac{x}{y} \times \dots}_{m \text{ times}} \\ &= \frac{x \times x \times x \times \dots m \text{ times}}{y \times y \times y \times \dots m \text{ times}} \\ &= \frac{x^m}{y^m} \end{aligned}$$

Hence,

$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}.$$

From the above formulae, we observe that

- (i) To multiply two numbers with same base and different exponents, we add the exponents to the same base. Thus, $2^3 \times 2^5 = 2^{3+5} = 2^8$
- (ii) When a positive integral power is raised to a positive integral power, the base is raised to the product of the two powers.

Thus, $(3^5)^4 = 3^{5 \times 4} = 3^{20}$.

- (iii) To divide two numbers each with a positive integral exponent and with same base, subtract the exponent of the denominator from the exponent of the numerator.

Thus, $\frac{5^7}{5^3} = 5^{7-3} = 5^4$

- (iv) The positive integral power of a product is the product of the powers of the factors.

Thus, $(35)^4 = (7 \times 5)^4 = 7^4 \times 5^4$

- (v) A positive integral power of a number expressed as a fraction is equal to power of the numerator divided by the power of denominator.

Thus, $\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$

6.3 ZERO EXPONENT

We know that $x^m \div x^n = x^{m-n}$

Let

Then, $m = n$
 $x^n \div x^n = x^{n-n} = x^0$
 $\therefore 1 = x^0$

Hence, $x^0 = 1$

Definition

Any non-zero base with an index zero is equal to 1 i.e. $a^0 = 1$ when a is any non-zero real number.

6.4 NEGATIVE EXPONENT

We know that

$$x^n = \underbrace{x \times x \times x \times \dots}_{n \text{ factors}} \text{ where } n \text{ is the positive integer}$$

Now we shall define negative exponent in terms of fundamental laws of exponents.

We know that

$$x^m \times x^n = x^{m+n}$$

Let

Then, $m = -n$
 $x^{-n} \times x^n = x^{-n+n} = x^0 = 1$
 $\therefore x^{-n} = \frac{1}{x^n}$

Definition

Any base with negative exponent is the reciprocal of the base with positive exponent i.e.

$$x^{-n} = \frac{1}{x^n} \text{ where } x \text{ is a non zero real number and } n \text{ is an integer.}$$

For example, $a^{-2} = \frac{1}{a^2}$, $\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \left(\frac{3}{2}\right)^3$

Example 1. Evaluate each of the following:

$$(i) 2^3 \times 2^4 \quad (ii) (2^3)^2 \quad (iii) \left(\frac{5}{2}\right)^4 \quad (iv) \left(\frac{3}{11}\right)^{-2}$$

Solution. (i) $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$
(ii) $(2^3)^2 = 2^{3 \times 2} = 2^6 = 64$

$$(iii) \left(\frac{5}{2}\right)^4 = \frac{5^4}{2^4} = \frac{625}{16}$$

$$(iv) \left(\frac{3}{11}\right)^{-2} = \frac{1}{\left(\frac{3}{11}\right)^2} = \left(\frac{11}{3}\right)^2 = \frac{11^2}{3^2} = \frac{121}{9}$$

Example 2. Evaluate each of the following:

$$(i) \left(\frac{5}{3}\right)^3 \times \left(\frac{9}{2}\right)^4 \quad (ii) \left(\frac{2}{3}\right)^2 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^2 \quad (iii) \left(-\frac{2}{5}\right)^5 \times \left(\frac{3}{4}\right)^2$$

Solution. (i)
$$\begin{aligned} \left(\frac{5}{3}\right)^3 \times \left(\frac{9}{2}\right)^4 &= \frac{5^3}{3^3} \times \frac{9^4}{2^4} \\ &= \frac{5^3 \times (3^2)^4}{3^3 \times 2^4} \\ &= \frac{5^3 \times 3^8}{3^3 \times 2^4} \\ &= \frac{5^3 \times 3^{8-3}}{2^4} \\ &= \frac{5^3 \times 3^5}{2^4} \\ &= \frac{125 \times 243}{16} \\ &= \frac{30375}{16} \end{aligned}$$

$$\begin{aligned} (ii) \left(\frac{2}{3}\right)^2 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^2 &= \frac{2^2}{3^2} \times \frac{2^{-3}}{5^{-3}} \times \frac{3^2}{5^2} \\ &= \frac{2^2}{3^2} \times \frac{5^3}{2^3} \times \frac{3^2}{5^2} \\ &= \frac{5^{3-2} \times 3^{2-2}}{2^{3-2}} \end{aligned}$$

$$= \frac{5^1 \cdot 3^0}{2^1}$$

$$= \frac{5 \times 1}{2} = \frac{5}{2}$$

$$(iii) \quad \left(-\frac{2}{5}\right)^5 \times \left(\frac{3}{4}\right)^2 = \frac{(-2)^5 \times 3^2}{5^5 \times 4^2} = \frac{(-32) \times 9}{3125 \times 16}$$

$$= \frac{-18}{3125}$$

Example 3. Simplify each of the following:

$$(i) \quad (x^3)^4 \times (a^4)^3$$

$$(ii) \quad (2^9)^3 \times 2^{13} - 2^{15} \times (2^5)^5$$

Solution. (i) $(x^3)^4 \times (a^4)^3 = x^{3 \times 4} \times a^{4 \times 3}$

$$= x^{12} \times a^{12}$$

$$= x^{12} a^{12}$$

$$= (xa)^{12}$$

$$(ii) \quad (2^9)^3 \times 2^{13} - 2^{15} \times (2^5)^5$$

$$= 2^{9 \times 3} \times 2^{13} - 2^{15} \times 2^{5 \times 5}$$

$$= 2^{27} \times 2^{13} - 2^{15} \times 2^{25}$$

$$= 2^{27+13} - 2^{15+25}$$

$$= 2^{40} - 2^{40}$$

$$= 0$$

EXERCISE 6.1

Evaluate each of the following:

$$1. \quad (i) \quad (3)^4 \times (2)^3$$

$$(ii) \quad \left(\frac{11}{12}\right)^3$$

$$(iii) \quad \left(-\frac{5}{7}\right)^3$$

$$2. \quad (i) \quad (2^3)^2$$

$$(ii) \quad (3^2)^3$$

$$(iii) \quad 5^5 \div 5^3$$

$$3. \quad (i) \quad \left(\frac{3}{7}\right)^{-3}$$

$$(ii) \quad (-x)^3 (-y)^2$$

$$(iii) \quad (x^5)^6 \times (y^6)^5$$

$$4. \quad \left(\frac{2}{11}\right)^4 \times \left(\frac{11}{3}\right)^2 \times \left(\frac{3}{2}\right)^3$$

$$5. \quad \left(\frac{1}{2}\right)^5 \times \left(-\frac{2}{3}\right)^4 \times \left(\frac{3}{5}\right)^{-1}$$

$$6. \quad (-5p^2q)(-3pq^2)$$

$$7. \quad 2^{2^3} + 2^{3^2}$$

$$8. \quad 2^{2^3} - 2^{3^2}$$

$$9. \quad 3^{98} \cdot 3^5 - 3^{70} \cdot 3^{33}$$

$$10. \quad 5^{3^2} \div 5^{2^3}$$

$$11. \quad (2^2)^3 - 2^{2^3}$$

6.5 FRACTIONAL EXPONENTS

We know that $16 = 4^2$ or 4 is a square root of 16. We write it as $\sqrt{16}$.

Similarly, $125 = 5^3$. Therefore 5 is a cube root of 125. We write it as $\sqrt[3]{125}$ or $\sqrt[3]{125}$.

This leads to the following definition.

Definition

If a is any real number and n a whole number (i.e. positive integer), then x is called the n th root of a if and only if $x^n = a$ and we write $x = a^{\frac{1}{n}}$ or $\sqrt[n]{a}$

By laws of indices,

$$\begin{aligned} a^{\frac{1}{2}} \times a^{\frac{1}{2}} &= a^{\frac{1}{2} + \frac{1}{2}} = a = \left(a^{\frac{1}{2}}\right)^2 \\ \therefore a^{\frac{1}{2}} &= \sqrt{a} \end{aligned}$$

Again, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = a = \left(a^{\frac{1}{3}}\right)^3$

$$\therefore a^{\frac{1}{3}} = \sqrt[3]{a}$$

Similarly, $a^{\frac{1}{4}} = \sqrt[4]{a}$

In general, we have

$$\begin{aligned} a^{\frac{1}{m}} \times a^{\frac{1}{m}} \times a^{\frac{1}{m}} \times a^{\frac{1}{m}} \dots m \text{ factors} \\ &= a^{\frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \dots} \\ &= a^{\frac{m}{m}} = a \\ \therefore a^{\frac{1}{m}} &= \sqrt[m]{a} \end{aligned}$$

Hence, $a^{\frac{1}{m}}$ is the m th root of a

Now $a^{\frac{n}{m}} \times a^{\frac{n}{m}} \times a^{\frac{n}{m}} \times \dots m \text{ factors}$

$$\begin{aligned} &= a^{\frac{n}{m} + \frac{n}{m} + \frac{n}{m} + \dots m \text{ times}} \\ &= a^{\frac{n \times m}{m}} = a^n \\ \therefore a^{\left(\frac{n}{m}\right)^m} &= a^n \\ \therefore a^{\frac{n}{m}} &= \sqrt[m]{a^n} \end{aligned}$$

Thus, in the case of fractional exponents, numerator represents the power and the denominator of the index represents the root.

If we perform the power operation first and root afterwards or root first and power afterwards, we get the same result.

i.e.

$$a^{\frac{n}{m}} = (a^n)^{\frac{1}{m}} = (a^{\frac{1}{m}})^n$$

For example,

$$(2^4)^{\frac{1}{2}} = (16)^{\frac{1}{2}} = 4$$

and

$$(2^{\frac{1}{2}})^4 = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} = 4$$

Thus, we have the following laws of indices, applicable to any index positive, negative or fractional :

$$1. x^m \times x^n = x^{m+n}$$

$$2. (x^m)^n = x^{mn}$$

$$3. x^m \div x^n = x^{m-n}$$

$$4. (xy)^m = x^m y^m$$

$$5. \left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$$

$$6. x^0 = 1$$

$$7. x^{-n} = \frac{1}{x^n}$$

$$8. x^{\frac{1}{m}} = \sqrt[m]{x}$$

$$9. x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}} = (\sqrt[m]{x})^n$$

Example 1. Evaluate each of the following:

$$(i) 81^{\frac{1}{4}}$$

$$(ii) (64)^{-\frac{2}{3}}$$

$$(iii) (125)^{-\frac{1}{3}}$$

$$(iv) \left(\frac{16}{25}\right)^{-\frac{3}{2}}$$

$$\text{Solution.} \quad (i) 81^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3^{4 \times \frac{1}{4}} = 3^1 = 3$$

$$(ii) (64)^{-\frac{2}{3}} = \frac{1}{(64)^{\frac{2}{3}}} = \frac{1}{[4^3]^{\frac{2}{3}}} = \frac{1}{4^{3 \times \frac{2}{3}}} \\ = \frac{1}{4^2} = \frac{1}{16}$$

$$(iii) (125)^{-\frac{1}{3}} = \frac{1}{(125)^{\frac{1}{3}}} = \frac{1}{(5^3)^{\frac{1}{3}}} \\ = \frac{1}{5^{3 \times \frac{1}{3}}} = \frac{1}{5}$$

$$(iv) \left(\frac{16}{25}\right)^{-\frac{3}{2}} = \frac{1}{\left(\frac{16}{25}\right)^{\frac{3}{2}}} = \left(\frac{25}{16}\right)^{\frac{3}{2}} \\ = \left[\left(\frac{5}{4}\right)^2\right]^{\frac{3}{2}} = \left(\frac{5}{4}\right)^{2 \times \frac{3}{2}} \\ = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$$

$$\text{Example 2. Simplify: } \frac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$$

Solution.

$$\frac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}} = \frac{(5^2)^{\frac{3}{2}} \times (3^5)^{\frac{3}{5}}}{(2^4)^{\frac{5}{4}} \times (2^3)^{\frac{4}{3}}} \\ = \frac{5^3 \times 3^3}{2^5 \times 2^4} = \frac{125 \times 27}{32 \times 16} = \frac{3375}{512}$$

Example 3. Evaluate: $\frac{1}{2^{-1}+3^{-1}} + \frac{1}{2^{-1}-3^{-1}}$

Solution.

$$\begin{aligned}\frac{1}{2^{-1}+3^{-1}} + \frac{1}{2^{-1}-3^{-1}} &= \frac{1}{\frac{1}{2}+\frac{1}{3}} + \frac{1}{\frac{1}{2}-\frac{1}{3}} \\&= \frac{1}{\frac{3+2}{6}} + \frac{1}{\frac{3-2}{6}} \\&= \frac{1}{\frac{5}{6}} + \frac{1}{\frac{1}{6}} \\&= \frac{6}{5} + 6 = \frac{36}{5}\end{aligned}$$

Example 4. Evaluate each of the following:

$$(i) \left(\frac{x^{-4}}{y^{-5}} \right)^{\frac{5}{4}} \quad (ii) \left(\frac{256 x^{16}}{81 y^4} \right)^{-\frac{3}{4}} \quad (iii) \left(\frac{\sqrt{2}}{\sqrt{3}} \right)^5 \left(\frac{6}{7} \right)^2$$

$$\begin{aligned}\text{Solution.} \quad (i) \quad \left(\frac{x^{-4}}{y^{-5}} \right)^{\frac{5}{4}} &= \left(\frac{y^5}{x^4} \right)^{\frac{5}{4}} = \frac{y^{\frac{5 \times 5}{4}}}{x^{\frac{4 \times 5}{4}}} \\&= \frac{y^{\frac{25}{4}}}{x^5} = \frac{\sqrt[4]{y^{25}}}{x^5}\end{aligned}$$

$$\begin{aligned}(ii) \quad \left(\frac{256 x^{16}}{81 y^4} \right)^{-\frac{3}{4}} &= \left(\frac{81 y^4}{256 x^{16}} \right)^{\frac{3}{4}} \\&= \left(\frac{3^4 \times y^4}{4^4 \times x^{16}} \right)^{\frac{3}{4}} \\&= \frac{3^3 \times y^3}{4^3 \times x^{12}} = \frac{27 y^3}{64 x^{12}}\end{aligned}$$

$$\begin{aligned}(iii) \quad \left(\frac{\sqrt{2}}{\sqrt{3}} \right)^5 \left(\frac{6}{7} \right)^2 &= \left(\frac{\frac{1}{2^{\frac{1}{2}}}}{\frac{1}{3^{\frac{1}{2}}}} \right)^5 \left(\frac{6}{7} \right)^2 = \frac{\frac{5}{2^{\frac{1}{2}}}}{\frac{5}{3^{\frac{1}{2}}}} \times \frac{3^2 \times 2^2}{7^2} \\&= \frac{\frac{5+2}{2^{\frac{1}{2}}}}{\frac{5-2}{3^{\frac{1}{2}}} \times 7^2} = \frac{\frac{9}{2^{\frac{1}{2}}}}{\frac{1}{3^{\frac{1}{2}}} \times 7^2}\end{aligned}$$

$$= \frac{2^4 \cdot 2^{\frac{1}{2}}}{\frac{1}{3^{\frac{1}{2}}} \times 49} = \frac{16\sqrt{2}}{49\sqrt{3}}$$

Example 5. Simplify :

$$\begin{aligned} & \left(\frac{x^l}{x^m} \right)^{l+m} \times \left(\frac{x^m}{x^n} \right)^{m+n} \times \left(\frac{x^n}{x^l} \right)^{n+l} \\ \text{Solution. } & \left(\frac{x^l}{x^m} \right)^{l+m} \times \left(\frac{x^m}{x^n} \right)^{m+n} \times \left(\frac{x^n}{x^l} \right)^{n+l} \\ & = (x^{l-m})^{l+m} \times (x^{m-n})^{m+n} \times (x^{n-l})^{n+l} \\ & = x^{(l-m)(l+m)} \times x^{(m-n)(m+n)} \times x^{(n-l)(n+l)} \\ & = x^{l^2 - m^2} \times x^{m^2 - n^2} \times x^{n^2 - l^2} \\ & = x^{l^2 - m^2 + m^2 - n^2 + n^2 - l^2} \\ & = x^0 = 1 \end{aligned}$$

Example 6. Simplify:

$$\begin{aligned} & \left(\frac{x^a}{x^b} \right)^{a^2 + ab + b^2} \times \left(\frac{x^b}{x^c} \right)^{b^2 + bc + c^2} \times \left(\frac{x^c}{x^a} \right)^{c^2 + ca + a^2} \\ \text{Solution. } & \left(\frac{x^a}{x^b} \right)^{a^2 + ab + b^2} \times \left(\frac{x^b}{x^c} \right)^{b^2 + bc + c^2} \times \left(\frac{x^c}{x^a} \right)^{c^2 + ca + a^2} \\ & = (x^{a-b})^{a^2 + ab + b^2} \times (x^{b-c})^{b^2 + bc + c^2} \times (x^{c-a})^{c^2 + ca + a^2} \\ & = x^{(a-b)(a^2 + ab + b^2)} \times x^{(b-c)(b^2 + bc + c^2)} \times x^{(c-a)(c^2 + ca + a^2)} \\ & = x^{a^3 - b^3} \times x^{b^3 - c^3} \times x^{c^3 - a^3} \\ & = x^{a^3 - b^3 + b^3 - c^3 + c^3 - a^3} \\ & = x^0 = 1 \end{aligned}$$

Example 7. If $27^x = \frac{9}{3^x}$, find the value of x .

Solution. We are given $27^x = \frac{9}{3^x}$

$$\text{or } (3^3)^x = \frac{9}{3^x}$$

$$\text{or } 3^{3x} \times 3^x = 9$$

$$\text{or } 3^{3x+x} = 3^2$$

$$\text{or } 3^{4x} = 3^2$$

As bases are same, exponents are equal

$$\therefore 4x = 2$$

or $x = \frac{1}{2}$

Example 8. If $25^{n-1} = 5^{2n-1} - 100$, find n .

Solution. We are given

$$\begin{aligned} & 25^{n-1} = 5^{2n-1} - 100 \\ \text{or } & (5^2)^{n-1} = 5^{2n-1} - 100 \\ \text{or } & 5^{2n-2} - 5^{2n-1} = -100 \\ \text{or } & 5^{2n-2} - 5^{2n-2+1} = -100 \\ \text{or } & 5^{2n-2} - 5^{2n-2} \times 5^1 = -100 \\ \text{or } & 5^{2n-2} [1 - 5] = -100 \\ \text{or } & 5^{2n-2} (-4) = -100 \\ \text{or } & 5^{2n-2} = \frac{-100}{-4} = 25 \\ \text{or } & 5^{2n-2} = 5^2 \\ \therefore & 2n - 2 = 2 \\ \text{or } & 2n = 4 \\ \therefore & n = 2 \end{aligned}$$

Example 9. If $a = b^{2x}$, $b = c^{2y}$ and $c = a^{2z}$, show that $xyz = \frac{1}{8}$.

Solution. We are given

$$\begin{aligned} a &= b^{2x} && \dots(i) \\ b &= c^{2y} && \dots(ii) \\ \text{and } & c = a^{2z} && \dots(iii) \end{aligned}$$

Substituting $b = c^{2y}$ from (ii) in (i), we get

$$a = (c^{2y})^{2x} = c^{4xy} \quad \dots(iv)$$

Substituting $c = a^{2z}$ from (iii) in (iv), we get

$$a^1 = (a^{2z})^{4xy} = a^{8xyz}$$

$$\therefore 1 = 8xyz$$

$$\text{or } xyz = \frac{1}{8}$$

Example 10. If $2^x = 7^{-y} = 14^z$, show that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Solution. We are given

$$\begin{aligned} 2^x &= 7^{-y} = 14^z \\ \text{Let } & 2^x = 7^{-y} = 14^z = k \text{ (say)} \end{aligned}$$

$$\therefore 2^x = k \text{ gives } 2 = k^{\frac{1}{x}} \quad \dots(i)$$

$$\text{Similarly, } 7^{-y} = k \text{ gives } 7 = k^{\frac{1}{y}} \quad \dots(ii)$$

$$\text{and } 14^z = k \text{ gives } 14 = k^{\frac{1}{z}}$$

$$\text{Now, } k^{\frac{1}{z}} = 14 = 2 \times 7 = k^{\frac{1}{x}} \times k^{\frac{1}{y}} \quad \dots[\text{Using (i) and (ii)}]$$

$$\text{or } k^{\frac{1}{z}} = k^{\frac{1}{x} - \frac{1}{y}}$$

$$\therefore \frac{1}{z} = \frac{1}{x} - \frac{1}{y}$$

$$\text{or } \frac{1}{x} = \frac{1}{y} + \frac{1}{z}$$

Example 11. If $p^x = q^y = r^z$ and $q^2 = pr$, then show that

$$y = \frac{2xz}{x+z}$$

Solution. We are given

$$p^x = q^y = r^z = k \text{ (say)}$$

$$\therefore p = k^{\frac{1}{x}}, q = k^{\frac{1}{y}} \text{ and } r = k^{\frac{1}{z}}$$

$$\text{Now, } q^2 = pr \text{ gives us } (k^{\frac{1}{y}})^2 = k^{\frac{1}{x}} \times k^{\frac{1}{z}}$$

$$\text{or } k^{\frac{2}{y}} = k^{\frac{1}{x} + \frac{1}{z}}$$

$$\therefore \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$$\text{or } \frac{2}{y} = \frac{z+x}{xz}$$

$$\text{Hence, } y = \frac{2xz}{z+x} \text{ or } \frac{2xz}{x+z}$$

Example 12. Solve the equation:

$$2^{2x+1} = 17 \cdot 2^x - 2^3$$

Solution. The given equation is

$$2^{2x+1} = 17 \cdot 2^x - 2^3$$

$$\text{or } 2^{2x+1} - 17 \cdot 2^x + 2^3 = 0$$

$$\text{or } 2^{2x} \cdot 2^1 - 17 \cdot 2^x + 8 = 0 \quad \dots(i)$$

$$\text{Let } 2^x = y, \text{ then } 2^{2x} = y^2$$

$\therefore (i)$ becomes

$$2y^2 - 17y + 8 = 0$$

$$\text{or } 2y^2 - 16y - y + 8 = 0 \quad \dots[-17 = (-16) + (-1) \text{ and } (-16)(-1) = 2 \times 8]$$

$$\text{or } 2y(y - 8) - 1(y - 8) = 0$$

$$\text{or } (y - 8)(2y - 1) = 0$$

$$\text{or } y - 8 = 0 \text{ or } 2y - 1 = 0$$

$$\therefore y = 8 \text{ or } y = \frac{1}{2}$$

$$\text{Now } y = 2^x = 8 = 2^3$$

$$\therefore x = 3$$

$$\text{and } y = 2^x = \frac{1}{2} = 2^{-1}$$

$$\therefore x = -1$$

$$\text{Hence, } x = 3 \text{ and } x = -1$$

Example 13. Solve the following system of equations for x and y :

$$3^x = 9 \times 3^y \text{ and } 8 \times 2^y = 4^x$$

Solution. The given equations are

$$3^x = 9 \times 3^y \quad \dots(i)$$

$$\text{and} \quad 8 \times 2^y = 4^x \quad \dots(ii)$$

Rewriting (i) and (ii),

$$3^x = 3^2 \times 3^y$$

$$\text{and} \quad 4^x = 2^3 \times 2^y$$

$$\text{or} \quad 3^x = 3^{2+y} \quad \dots(iii)$$

$$\text{and} \quad 2^{2x} = 2^{y+3} \quad \dots(iv)$$

From (iii) and (iv), we get

$$x = 2 + y$$

$$\text{and} \quad 2x = y + 3$$

$$\text{or} \quad x - y - 2 = 0 \quad \dots(v)$$

$$\text{and} \quad 2x - y - 3 = 0 \quad \dots(vi)$$

Subtracting (v) from (vi), we get

$$x - 1 = 0$$

$$\text{or} \quad x = 1$$

∴ From (v),

$$1 - y - 2 = 0$$

$$\text{or} \quad y = -1$$

Hence, $x = 1$ and $y = -1$

Example 14. Prove that

$$\frac{2^{m+1} \times 3^{2m-n} \times 5^{m+n} \times 6^n}{6^m \times 10^{n+2} \times 15^m} = \frac{1}{50}$$

$$\text{L.H.S.} = \frac{2^{m+1} \times 3^{2m-n} \times 5^{m+n} \times 6^n}{6^m \times 10^{n+2} \times 15^m}$$

$$\begin{aligned} &= \frac{2^{m+1} \times 3^{2m-n} \times 5^{m+n} \times 2^n \times 3^n}{2^m \times 3^m \times 2^{n+2} \times 5^{n+2} \times 5^m \times 3^m} \\ &= 2^{m+1+n-m-(n+2)} \times 3^{2m-n+n-m-m} \times 5^{m+n-(n+2)-m} \\ &= 2^{-1} \times 3^0 \times 5^{-2} \end{aligned}$$

$$= \frac{1}{2} \times 1 \times \frac{1}{25}$$

$$= \frac{1}{50} = \text{R.H.S.}$$

EXERCISE 6.2

Evaluate each of the following (1–4) :

1. (i) $9^{-\frac{1}{2}}$

(ii) $27^{-\frac{1}{3}}$

(iii) $\sqrt[5]{243x^{10}y^5z^{10}}$

2. (i) $\left(16^{-\frac{1}{5}}\right)^{5/2}$

(ii) $(6^2)^3 \div 6^{2^3}$

(iii) $\frac{2^{n+1}-2^n}{2^{n+2}+2^{n+1}}$

3. (i) $2^{2^{2^2}} - [(2^2)^2]^3$

(ii) $\left[8^{-\frac{4}{3}} \div 2^{-2}\right]^{1/2}$

(iii) $9^{5/2} - 3 \times (5)^0 \times \left(\frac{1}{81}\right)^{-\frac{1}{2}}$

4. (i) $\frac{(4 \times 10^7)(6 \times 10^{-5})}{8 \times 10^{10}}$

(ii) $\sqrt[3]{(16)^{-3/4} \times (125)^{-2}}$

(iii) $\left[\left(\sqrt[4]{x^{-3/4}}\right)^{-\frac{4}{3}}\right]^4$

Simplify each of the following (5 – 18) :

5. $(x^a - b)^{a+b} \times (x^b - c)^{b+c} \times (x^c - a)^{c+a}$

6. $\left(\frac{x^l}{x^m}\right)^{l+m} \times \left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^l}\right)^{n+l}$

7. $\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \times \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\frac{x^c}{x^a}\right)^{c^2+ac+a^2}$

8. $\left(\frac{x^{a^2+b^2}}{x^{ab}}\right)^{a+b} \times \left(\frac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \times \left(\frac{x^{c^2+a^2}}{x^{ca}}\right)^{c+a}$

9. $(x^{l+m})^{l^2+m^2-lm} \times (x^{m+n})^{m^2+n^2-mn} \times (x^{n+l})^{n^2+l^2-nl}$

10. $\left(\frac{x^{l+m}}{x^n}\right)^{l-m} \times \left(\frac{x^{m+n}}{x^l}\right)^{m-n} \times \left(\frac{x^{l+n}}{x^m}\right)^{n-l}$

11. $\frac{1}{1+x^{a+b}} + \frac{1}{1+x^{b-a}}$

12. $\frac{1}{1+x^{b-a}+x^{c-a}} + \frac{1}{1+x^{a-b}+x^{c-b}} + \frac{1}{1+x^{b-c}+x^{a-c}}$

13. $\frac{2^{x+1} \times 3^{4x} \times 5^{x+y} \times 25^{2x+1}}{81^x \times 4^{4x+y} \times 10^x}$

14. $\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$

15. $\frac{6(8)^{n+1} + 16(2)^{3n-2}}{10(2)^{3n+1} - 7(8)^n}$

16. $\frac{5^{2n+3} - (25)^{n+2}}{[(125)^{n+1}]^{2/3}}$

17. $\frac{32^{\frac{2}{5}} \times 4^{-\frac{1}{2}} \times 8^{\frac{1}{3}}}{2^{-2} \div (64)^{-\frac{1}{3}}}$

18. $\frac{16 \times 2^{m+1} - 4 \times 2^m}{16 \times 2^{m+2} - 2 \times 2^{m+2}}$

For what value of x , each of the following is true (19 – 24) ? :

19. $a^x \cdot a^{4x} = a^{20}$

20. $3^{x+2} = 3^{2x-4}$

21. $\left(\frac{3}{5}\right)^x \left(\frac{5}{3}\right)^{2x} = \frac{125}{27}$

22. $5^{2x+1} = 5^{x+3}$

23. $2^{5x} \div 2^x = \sqrt[5]{2^{20}}$

24. $(2^3)^4 = (2^2)^x$

25. If $5^{x-2} \cdot 3^{2x-3} = 135$, find x

26. If $2^{a-5} \cdot 5^{a-4} = 5$, find a

27. If $3^{2m+5} = 3^{2m+3} + 72$, find m

28. If $2^{y-7} \times 5^{y-6} = 50$, find y

29. If $x = 2$ and $y = 3$, find the value of

(i) $x^x + y^y$

(ii) $\left(\frac{y}{x}\right)^y$

(iii) $x^y + y^x$

(iv) $\left(\frac{1}{x} + \frac{1}{y}\right)^y$

30. Prove that $\frac{x^{-1}}{x^{-1}+y^{-1}} + \frac{x^{-1}}{x^{-1}-y^{-1}} = \frac{2y^2}{y^2-x^2}$

31. Prove that $\left(\frac{x^a}{x^b}\right)^{\frac{1}{ab}} \times \left(\frac{x^b}{x^c}\right)^{\frac{1}{bc}} \times \left(\frac{x^c}{x^a}\right)^{\frac{1}{ca}} = 1$

32. If $a^x = b$, $b^y = c$ and $c^z = a$, $a \neq 0$, then prove that $xyz = 1$

33. If $2^x = 3^y = 6^z$, then prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

34. If $2^x = 3^y = 12^z$, then show that $x = \frac{2yz}{y-z}$

35. Solve the equation :

$$3(2^x + 1) - 2^{x+2} + 5 = 0$$

36. If $\frac{9^n \times 3^{n+2} - (27)^n}{3^{3m} \times 2^3} = \frac{1}{27}$, show that $n = m - 1$

37. Solve for x :

$$\sqrt{\left(8^0 + \frac{2}{3}\right)} = (0.6)^{2-3x} \quad (\text{I.C.S.E.})$$

38. If $x = ab^{p-1}$, $y = ab^{q-1}$ and $z = ab^{r-1}$, then show that $x^{q-r} \times y^{r-p} \times z^{p-q} = 1$

Let us Sum up

- An exponent (index) is a number which indicates how many times another number, the base is being used as repeated factor.

- Laws of exponents are:

- (i) $x^m \times x^n = x^{m+n}$

- (ii) $(x^m)^n = x^{mn}$

- (iii) $x^m \div x^n = x^{m-n}$, $m > n$

and $x^m \div x^n = \frac{1}{x^{n-m}}$, $m < n$

(iv) $(xy)^m = x^m y^m$ and $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

- $x^0 = 1$

- $x^{-n} = \frac{1}{x^n}$

- If a is any real number and n a whole number, then x is called the n th root of a if and only if $x^n = a$.

- n th root of a is denoted by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$.

Check Your Progress-6

1. Evaluate : $5^0 \times 4^{-1} + 8^{1/3}$ (I.C.S.E.)
2. Evaluate : $\left(\frac{64}{125}\right)^{-2/3} \div \frac{1}{\left(\frac{256}{625}\right)^{1/4}} + \left(\frac{\sqrt{25}}{\sqrt[3]{64}}\right)^0$ (I.C.S.E.)
3. Evaluate : $9^{3/2} - 3 \times 5^0 - \left(\frac{1}{81}\right)^{-\frac{1}{2}}$ (I.C.S.E.)
4. Evaluate : $\left[\frac{1}{4}\right]^{-2} - 3(8)^{2/3} \times 4^0 + \left(\frac{9}{16}\right)^{-\frac{1}{2}}$ (I.C.S.E.)
5. Evaluate : $16^{3/4} + 2\left(\frac{1}{2}\right)^{-1} \times 3^0$ (I.C.S.E.)
6. Evaluate : $(81)^{3/4} - \left(\frac{1}{32}\right)^{-\frac{2}{5}} + (8)^{1/3} \left[\frac{1}{2}\right]^{-2} (2)^0$ (I.C.S.E.)
7. If $x = 9$, $y = 2$ and $z = 8$, then evaluate $x^{\frac{1}{2}} \times y^{-1} \times z^{2/3}$ (I.C.S.E.)
8. Evaluate as a fraction : $\left(\frac{27}{8}\right)^{\frac{2}{3}} - \left(\frac{1}{4}\right)^{-2} + 5^0$ (I.C.S.E.)
9. Solve the equation : $\left(\sqrt[3]{5}\right)^{x+1} = \frac{125}{27}$ (I.C.S.E.)
10. Find the value of x if $\sqrt[3]{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27}$ (I.C.S.E.)
11. Solve for x : $\left(\sqrt[3]{4}\right)^{2x+\frac{1}{2}} = \frac{1}{32}$ (I.C.S.E.)
12. Find the value of x if $\sqrt{\frac{p}{q}} = \left(\frac{q}{p}\right)^{1-2x}$ (I.C.S.E.)
13. Solve for x : $2^3(5^0 + 3^{2x}) = 8\frac{8}{27}$ (I.C.S.E.)

1. Simplify: $(32)^{\frac{4}{5}} + \left(\frac{1}{81}\right)^{-\frac{3}{4}} - \left(\frac{1}{125}\right)^{-2/3} - 6^0 \times 16^{\frac{3}{2}}$

2. If $x^a = y, y^b = z, z^c = x$, prove that $abc = 1$

3. Show that

$$\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2} \times \left(\frac{x^n}{x^l}\right)^{n^2+ln+l^2} \times \left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2} = 1$$

4. If $\frac{49^{n+1} \times 7^n - (343)^n}{7^{3m} \times 2^4} = \frac{3}{343}$, show that $m - n = 1$

5. If $3^x = 5^y = 45^z$, show that $x = \frac{2yz}{y-z}$

6. Solve the equation: $2^{5x+3} = 8^{x+3}$

7. Solve the equations: $(25)^{x-1} = 5^{y+1}$ and $\left(\frac{1}{3}\right)^{3-x} = \left(\frac{1}{9}\right)^{2y}$

8. Simplify:
$$\frac{8^m \times 15^{m+n-1} \times 16^{2m-n}}{6^{m+n} \times 4^{m-1}}$$

ANSWERS

Exercise 6.1

1. (i) 648 (ii) $\frac{1331}{1728}$ (iii) $\frac{-125}{343}$ 2. (i) 64 (ii) 729 (iii) 25
 3. (i) $\frac{343}{27}$ (ii) $-x^3y^2$ (iii) $(xy)^{30}$ or $x^{30}y^{30}$ 4. $\frac{6}{121}$ 5. $\frac{5}{486}$
 6. $15 p^3q^3$ 7. 320 8. -256 9. 0 10. 5
 11. -192

Exercise 6.2

1. (i) $\frac{1}{3}$ (ii) $\frac{1}{3}$ (iii) $3x^2yz^2$ 2. (i) $\frac{1}{4}$ (ii) $\frac{1}{36}$ (iii) $\frac{1}{6}$
 3. (i) 61440 (ii) $\frac{1}{2}$ (iii) 231 4. (i) $\frac{3}{10^8}$ (ii) $\frac{1}{50}$ (iii) x
 5. 1. 6. 1 7. 1 8. $x^{2(a^3+b^3+c^3)}$ 9. $x^{2(l^3+m^3+n^3)}$
 10. 1 11. 1 12. 1 13. $2^{1-8x-2y} \times 5^{4x+y+2}$ 14. $\frac{xy}{x+y}$
 15. 4 16. -20 17. 4 18. $\frac{1}{2}$ 19. 4
 20. 6 21. 3 22. 2 23. 1 24. 6
 25. $x = 3$ 26. $a = 5$ 27. $m = \frac{-1}{2}$ 28. $y = 8$
 29. (i) 31 (ii) $\frac{27}{8}$ (iii) 17 (iv) $\frac{125}{216}$ 35. $x = 3$
 37. $x = \frac{5}{6}$

Check Your Progress-6

1. $\frac{9}{4}$ 2. $\frac{9}{4}$ 3. 15 4. $\frac{16}{3}$ 5. 12
 6. 31 7. 6 8. $-\frac{51}{4}$ 9. $x = -7$ 10. $x = \frac{7}{2}$
 11. $x = -4$ 12. $x = \frac{3}{4}$ 13. $x = \frac{-3}{2}$

Self Assessment Test-6

1. -46 6. $x = 3$ 7. $x = \frac{5}{3}, y = \frac{1}{3}$ 8. $\frac{2^{2m-2n+2} \times 5^{3m-1}}{3}$